*J. Fluid Mech.* (2006), *vol.* 551, *pp.* 303–308. © 2006 Cambridge University Press 303
doi:10.1017/S0022112005008554 Printed in the United Kingdom

# Effects of wall suction/injection on the linear stability of flat Stokes layers

By PENG GAO AND XI-YUN LU†
Department of Modern Mechanics, University of Science and Technology of China, Hefei, Anhui 230026, P. R. China

(Received 24 September 2005 and in revised form 7 December 2005)

Effects of wall suction/injection on the linear stability of flat Stokes layers are investigated. A semi-analytical method is used to examine the stability of time-periodic boundary flows with wall suction/injection. Results show that the onset of instability of the flat Stokes layers can be suppressed by wall suction and enhanced by wall injection.

## 1. Introduction

This paper presents a study of the effect of a uniform and steady wall suction/injection on the linear stability of flat Stokes layers. The Stokes layer, induced by an infinite rigid plate oscillating sinusoidally in its own plane, is of interest as a prototype problem for time-periodic flows. Wall suction/injection has been widely used as an effective way to control flows (e.g. Joslin 1998; Wu *et al.* 1998), and its effect on the stability of unsteady boundary flows is of significant interest.

A linear stability analysis of flat Stokes layers was first carried out by von Kerczek & Davis (1974) by numerically integrating the time-dependent linearized disturbance equations based on the Floquet theory for a ‘finite Stokes layer’, in which a stationary upper infinite plate was introduced. For Reynolds number $R$ based on the thickness of the Stokes layer $\delta$ less than 800, results showed that the principal Floquet exponent was real and negative, indicating that the flow was stable to the disturbances. Hall (1978) performed a stability analysis of an infinite Stokes layer using a semi-analytical solution of the time-dependent Orr–Sommerfeld equation. Due to the limit of computer power, however, Hall only undertook calculations for $R < 320$. The results demonstrated the existence of a damped continuous spectrum of the Orr–Sommerfeld equation and a set of discrete eigenvalues for certain values of $R$. Although Hall speculated that the discrete Floquet modes could be unstable for sufficiently large Reynolds numbers, no evidence of instability was detected for the parameters considered. Later, Akhavan, Kamm & Shapiro (1991*b*) also failed to find growing Floquet modes for $R$ up to 1000. Since it has been found experimentally that turbulence emerges for $R > 550$ (Hino, Sawamoto & Takasu 1976; Akhavan, Kamm & Shapiro 1991*a*), the inconsistency between theoretical and experimental results has motivated study of alternative mechanisms, such as quasi-steady theory (Obremski & Morkovin 1969; Cowley 1987; Hall 2003) and nonlinear analysis (Wu & Cowley 1995). Recently, Blennerhassett & Bassom (2002, hereafter referred to as BB) reformulated the technique used by Hall (1978) and performed calculations to higher Reynolds number. They found a critical Reynolds number of about 1416 for disturbance with a wavenumber of $0.38/\delta$.

† Author to whom correspondence should be addressed: xlu@ustc.edu.cn

Furthermore, it is well known that the asymptotic suction boundary layer is much more stable than the Blasius boundary layer (Hughes & Reid 1965; Hocking 1975; Fransson & Alfredsson 2003). Generally, wall suction/injection is associated with stabilizing/destabilizing the flow. To the best of our knowledge, however, the effects of wall suction/injection on the stability of unsteady flows have never been studied. Here, we extend Hall's semi-analytical method to study the stability characteristics of a Stokes layer subject to wall suction/injection, especially the variation of the critical Reynolds number for different suction/injection rates.

## 2. Formulation and numerical procedure

Consider the flow of a semi-infinite viscous fluid of viscosity $\nu$ induced by a flat plate at $y = 0$ and oscillating with velocity $U_0 \cos \omega t$ in the $x$-direction. On the wall a uniform normal suction/injection velocity $V_0$ is prescribed. We take $\delta = \sqrt{2\nu/\omega}$ as the scale of length, $U_0$ as the scale of velocity, and introduce the non-dimensional time $\tau = \omega t$. There exist two Reynolds numbers in this problem, which are defined by

$$R = U_0\sqrt{\frac{2}{\nu\omega}}, \quad R_v = V_0\sqrt{\frac{2}{\nu\omega}}.$$

The sign of $R_v$ prescribes either a suction condition ($R_v < 0$) or injection ($R_v > 0$). The dimensionless form of the basic flow is then given by

$$U = \mathrm{e}^{-Py}\cos(\tau - Qy), \quad V = R_v/R, \tag{2.1}$$

where

$$P(R_v) = \sqrt{\frac{\sqrt{R_v^4+64}+R_v^2}{8}} - \frac{R_v}{2}, \quad Q(R_v) = \sqrt{\frac{\sqrt{R_v^4+64}-R_v^2}{8}}.$$

Note that $U$ only depends on $R_v$ and the basic flow of a flat Stokes layer is retrieved when $R_v = 0$.

According to the procedure proposed by von Kerczek & Davis (1974), it is easy to prove that, for a fixed $R_v$, the basic state (2.1) first becomes unstable for two-dimensional disturbances. Hence, only two-dimensional disturbances are considered. The basic flow (2.1) is infinitesimally disturbed to

$$(u, v) = (U, V) + \epsilon\left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x}\right), \tag{2.2}$$

where $\epsilon \ll 1$ and $\Psi$ denotes a disturbance stream function. Since the basic flow is periodic in time and independent of $x$, we can introduce a wave-like disturbance

$$\Psi(x, y, \tau) = \mathrm{e}^{\mu\tau}\mathrm{e}^{\mathrm{i}ax}\psi(y, \tau) + \text{c.c.} \tag{2.3}$$

where c.c. denotes complex conjugate, $a$ is the real streamwise wavenumber, and the Floquet exponent $\mu = \mu_r + \mathrm{i}\mu_i$ is complex. Function $\psi(y, \tau)$ is $2\pi$-periodic in time and satisfies

$$\left(2\frac{\partial}{\partial \tau} + 2\mu + \mathrm{i}aRU + R_v\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial y^2} - a^2\right)\psi = \left(\frac{\partial^2}{\partial y^2} - a^2\right)^2\psi + \mathrm{i}aRU_{yy}\psi. \tag{2.4a}$$

The boundary conditions are

$$\psi = \psi_y = 0 \quad \text{on} \quad y = 0; \quad \psi, \psi_y \to 0 \quad \text{as} \quad y \to \infty. \tag{2.4b}$$

The problem (2.4) has a symmetry property that if $\mu$, $\psi(y,\tau)$ is a solution, then so is $\tilde{\mu}$, $\tilde{\psi}(y,\tau+\pi)$, where a tilde denotes complex conjugate. Physically, if there exists a disturbance wave propagating in one direction, another wave propagating in the opposite direction with the same growth rate may also occur. This property allows us to restrict the imaginary part of $\mu$ to the interval $0 \leqslant \mu_i \leqslant \frac{1}{2}$ without loss of generality.

As $\psi$ is a periodic function of $\tau$, it can be expanded in a Fourier series

$$\psi(y,\tau) = \sum_{n=-\infty}^{\infty} \psi_n(y)\mathrm{e}^{\mathrm{i}n\tau}. \tag{2.5}$$

Substituting (2.5) into (2.4) leads to the following system of equations:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - R_v\frac{\mathrm{d}}{\mathrm{d}y} - a^2 - 2\mu - 2\mathrm{i}n\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - a^2\right)\psi_n$$
$$= \tfrac{1}{2}\mathrm{i}aR\left\{\mathrm{e}^{-C_p y}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - a^2 - C_p^2\right)\psi_{n-1} + \mathrm{e}^{-C_m y}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - a^2 - C_m^2\right)\psi_{n+1}\right\} \tag{2.6}$$

where $C_p = P + \mathrm{i}Q$ and $C_m = P - \mathrm{i}Q$.

As in Seminara & Hall (1976) and Hall (1978), a solution of (2.6) in a double series expansion can be derived as

$$\psi_n(y) = \sum_{k=-\infty}^{\infty}\left\{\alpha_k\sum_{j=0}^{\infty} A_{jkn}\mathrm{e}^{-f(j,k,n,\gamma_k)y} + \beta_k\sum_{j=0}^{\infty} B_{jkn}\mathrm{e}^{-f(j,k,n,a)y}\right\}. \tag{2.7}$$

Here, coefficients $A_{jkn}$ and $B_{jkn}$ can be determined by the recurrence relations given in the Appendix, $f$ takes the form

$$f(j,k,n,\cdot) = \cdot - \mathrm{i}(k-n)Q + |k-n|P + 2jP, \tag{2.8}$$

and

$$\gamma_k = \sqrt{R_v^2/4 + a^2 + 2\mu + 2\mathrm{i}k} - R_v/2, \tag{2.9}$$

where the branch of the square root with positive real part is taken.

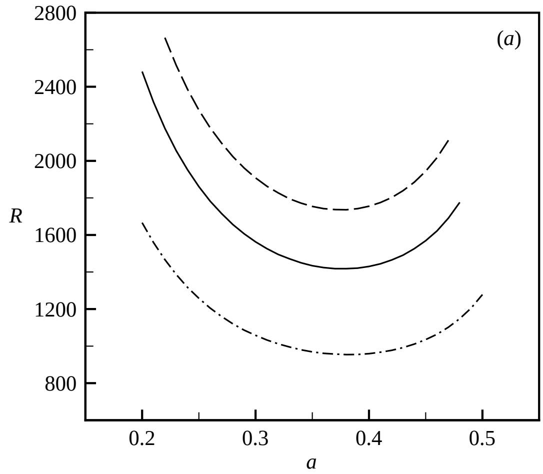
The solution (2.7) decays exponentially and satisfies automatically the boundary conditions as $y$ approaches infinity. The boundary conditions at $y = 0$ require

$$\sum_{k=-\infty}^{\infty}\left\{\alpha_k\sum_{j=0}^{\infty} A_{jkn} + \beta_k\sum_{j=0}^{\infty} B_{jkn}\right\} = 0, \tag{2.10a}$$

$$\sum_{k=-\infty}^{\infty}\left\{\alpha_k\sum_{j=0}^{\infty} A_{jkn} f(j,k,n,\gamma_k) + \beta_k\sum_{j=0}^{\infty} B_{jkn} f(j,k,n,a)\right\} = 0, \tag{2.10b}$$

for $n = 0, \pm 1, \pm 2, \ldots$ implying an infinite set of equations for the unknowns $\alpha_k$ and $\beta_k$. If the system has a non-trivial solution, the infinite determinant of the matrix of the coefficients must vanish. Then, the eigenrelation for $\mu$ in terms of $a$, $R$ and $R_v$ can be obtained.

Our numerical treatment follows BB and is outlined here. First, the Fourier series (2.5) are truncated to $N$ leading harmonics, and the index $k$ in (2.7) is limited to the range $-N \leqslant k \leqslant N$. The summations over $j$ in (2.10) are taken to a finite value $J$. The determinant of the matrix of (2.10) is of order $4N+2$ and is evaluated by $LU$-factorization. Muller's iteration method is used to locate the zeros of the determinant. After the eigenvalue $\mu$ is determined, the corresponding eigenfunction



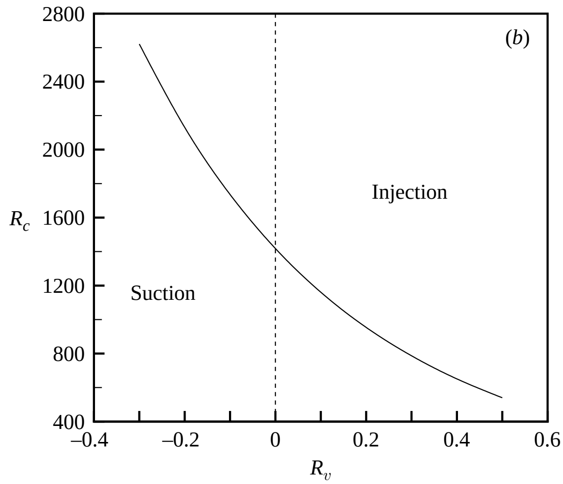




FIGURE 1. ($a$) Neutral stability curves for three values of $R_v$: – – – – –, $R_v = -0.1$; ———, $R_v = 0$; — · —, $R_v = 0.2$. ($b$) The critical Reynolds number $R_c$ as a function of $R_v$ in the range $-0.3 \leqslant R_v \leqslant 0.5$.

$[\alpha_{-N}, \dots, \alpha_N, \beta_{-N}, \dots, \beta_N]$ can be solved via inverse iteration. Then the disturbance flow field is reconstructed by the eigenfunction. To obtain the neutral curve, 128-bit arithmetic is needed to retain sufficient accuracy.

## 3. Results and discussion

To ensure the reliability of the present results, extensive calculations for the stability characteristics of flat Stokes layers without suction/injection (i.e. $R_v = 0$) were carried out. The results, including critical Reynolds number, growth rate, neutral curve, disturbance stream function and enstrophy, agree well with those of BB. We now present typical results on the effect of wall suction/injection on the stability.

Figure 1($a$) shows the neutral curves for $R_v = -0.1$, 0 and 0.2, obtained by interpolation through a series of neutral points with an increment of $\Delta a = 0.01$. The neutral curve shifts upwards for the suction and downwards for the injection. Although the influence of the suction/injection on the critical wavenumber is somewhat weak, it is still found that the effect of the wall suction tends to decrease the critical wavenumber, while the effect of the injection is to increase the critical wavenumber.

The critical Reynolds number, $R_c$, is shown in figure 1(*b*) for $-0.3 \leqslant R_v \leqslant 0.5$, indicating that $R_c$ monotonically decreases with $R_v$. Note that the normal velocity $V$ is weak compared to the streamwise basic flow near the wall; however, the stabilizing effect of the wall suction is remarkable so that the critical Reynolds number for $R_v = -0.3$ is about double of that for $R_v = 0$. To identify the critical Reynolds number for $R_v < -0.3$, truncation with $N > 400$ is needed, resulting in the calculation being expansive. Although the results for $R_v < -0.3$ are not shown here, we infer from figure 1(*b*) that the increase of the amplitude of suction would further stabilize the flow. Correspondingly, the flow is destabilized due to the effect of the wall injection, and $R_c$ decreases monotonically with increasing $R_v$.

## 4. Conclusion

A relatively weak wall suction/injection has a significant effect on the linear stability of Stokes layers. The results obtained by an extended formulation of Hall's (1978) semi-analytical method show that the neutral stability curve shifts upwards for wall suction and downwards for wall injection. The critical Reynolds number $R_c$ is a monotonic decreasing function of $R_v$, indicating a stabilizing effect of the suction and a destabilizing effect of the injection.

This work was supported by the National Natural Science Foundation of China (Nos. 90405007, 10125210), Specialized Research Fund for the Doctoral Program of Higher Education (No. 20020358013), and the Hundred Talents Program of the Chinese Academy of Sciences. The authors thank Professor A. P. Bassom for valuable comments and Professor J.-Z. Wu for the improvement of this work.

## Appendix. Recurrence relations

For convenience, define

$$F_p(j,k,n,\cdot) = \left[f^2(j,k,n,\cdot) - a^2 - C_p^2\right] A_{jkn}, \tag{A 1a}$$

$$F_m(j,k,n,\cdot) = \left[f^2(j,k,n,\cdot) - a^2 - C_m^2\right] A_{jkn}, \tag{A 1b}$$

$$G_p(j,k,n,\cdot) = \left[f^2(j,k,n,\cdot) - a^2 - C_p^2\right] B_{jkn}, \tag{A 1c}$$

$$G_m(j,k,n,\cdot) = \left[f^2(j,k,n,\cdot) - a^2 - C_m^2\right] B_{jkn} \tag{A 1d}$$

and

$$\phi(j,k,n,\cdot) = [f^2(j,k,n,\cdot) - a^2][f^2(j,k,n,\cdot) + R_v f(j,k,n,\cdot) - a^2 - 2\mu - 2\mathrm{i}n]. \tag{A 2}$$

If we substitute (2.7) for $\psi_n$ into (2.6) and equate the coefficients of the linearly independent terms, then the recurrence relations can be obtained. The coefficients $A_{jkn}$ and $B_{jkn}$ are scaled so that $A_{0kk} = B_{0kk} = 1$ and the remainder coefficients are determined as follows:

if $k \leqslant n-1$,

$$\phi(j,k,n,\gamma_k)A_{jkn} = \tfrac{1}{2}\mathrm{i}aR[F_p(j,k,n-1,\gamma_k) + F_m(j-1,k,n+1,\gamma_k)], \tag{A 3a}$$

$$\phi(j,k,n,a)B_{jkn} = \tfrac{1}{2}\mathrm{i}aR[G_p(j,k,n-1,a) + G_m(j-1,k,n+1,a)]; \tag{A 3b}$$

if $k \geqslant n+1$,

$$\phi(j,k,n,\gamma_k)A_{jkn} = \tfrac{1}{2}\mathrm{i}aR[F_p(j-1,k,n-1,\gamma_k) + F_m(j,k,n+1,\gamma_k)], \qquad \text{(A 3}c\text{)}$$

$$\phi(j,k,n,a)B_{jkn} = \tfrac{1}{2}\mathrm{i}aR[G_p(j-1,k,n-1,a) + G_m(j,k,n+1,a)]; \qquad \text{(A 3}d\text{)}$$

and if $k = n$,

$$\phi(j,k,n,\gamma_k)A_{jkn} = \tfrac{1}{2}\mathrm{i}aR[F_p(j-1,k,n-1,\gamma_k) + F_m(j-1,k,n+1,\gamma_k)], \qquad \text{(A 3}e\text{)}$$

$$\phi(j,k,n,a)B_{jkn} = \tfrac{1}{2}\mathrm{i}aR[G_p(j-1,k,n-1,a) + G_m(j-1,k,n+1,a)], \qquad \text{(A 3}f\text{)}$$

for $j = 0, 1, 2, \ldots$.

## REFERENCES


Akhavan, R., Kamm, R. D. & Shapiro, A. H. 1991*a* An investigation of transition to turbulence in bounded oscillatory Stokes flows. Part 1. Experiments. *J. Fluid Mech.* **225**, 395–422.

Akhavan, R., Kamm, R. D. & Shapiro, A. H. 1991*b* An investigation of transition to turbulence in bounded oscillatory Stokes flows. Part 2. Numerical simulations. *J. Fluid Mech.* **225**, 423–444.

Blennerhassett, P. J. & Bassom, A. P. 2002 The linear stability of flat Stokes layers. *J. Fluid Mech.* **464**, 393–410 (referred to herein as BB).

Cowley, S. J. 1987 High frequency Rayleigh instability analysis of Stokes layers. In *Stability of Time-dependent and Spatially Varying Flows* (ed. D. L. Dwoyer & M. Y. Hussaini), pp. 261–275. Springer.

Fransson, J. H. M. & Alfredsson, P. H. 2003 On the disturbance growth in an asymptotic suction boundary layer. *J. Fluid Mech.* **482**, 51–90.

Hall, P. 1978 The linear stability of flat Stokes layers. *Proc. R. Soc. Lond.* A **359**, 151–166.

Hall, P. 2003 On the instability of Stokes layers at high Reynolds numbers. *J. Fluid Mech.* **482**, 1–15.

Hino, M., Sawamoto, M. & Takasu, S. 1976 Experiments on transition to turbulence in an oscillatory pipe flow. *J. Fluid Mech.* **75**, 193–207.

Hocking, L. M. 1975 Non-linear instability of the asymptotic suction velocity profile. *Q. J. Mech. Appl. Maths* **28**, 341–353.

Hughes, T. H. & Reid, W. H. 1965 On the stability of the asymptotic suction boundary-layer profile. *J. Fluid. Mech.* **23**, 715–735.

Joslin, R. D. 1998 Aircraft laminar flow control. *Annu. Rev. Fluid Mech.* **30**, 1–29.

von Kerczek, C. & Davis, S. H. 1974 Linear stability theory of oscillatory layers. *J. Fluid Mech.* **62**, 753–773.

Obremski, H. J. & Morkovin, M. V. 1969 Apllication of a quasi-stedy stability model to periodic boundary-layer flows. *AIAA J.* **7**, 1298–1301.

Seminara, G. & Hall, P. 1976 Centrifugal instablity of a Stokes layer. *Proc. R. Soc. Lond.* A **350**, 299–316.

Wu, J. Z., Lu, X. Y., Denny, A. G., Fan, M. & Wu, J. M. 1998 Post-stall flow control on an airfoil by local unsteady forcing. *J. Fluid Mech.* **371**, 21–58.

Wu, X. S. & Cowley, S. J. 1995 On the nonlinear evolution of instabilty modes in unsteady shear layers–the Stokes layer as a paradigm. *Q. J. Mech. Appl. Maths* **48**, 159–188.